Journal of Mathematical Analysis and Applications **248,** 678–682 (2000)
doi:10.1006/jmaa.2000.6970, available online at http://www.idealibrary.com on IDEAL®

NOTE

A Note on the Extension Principle

Brafova 7, CZ-701 03 Ostrava, Czech Republic; and Department of Computer Science, Technical University of Ostrava, 17 Listopadu, CZ-700 30 Ostrava-Poruba, Czech Republic
E-mail: radim.belohlavek@osu.cz

Submitted by Ulrich Höhle

Received November 8, 1999

It is shown that a function obtained by Zadeh's extension principle preserves similarity. As a consequence, if using the extension principle for building models of systems, the exact description of phenomena by means of fuzzy sets does not matter.

1. INTRODUCTION

A fuzzy set in a universe set X is a mapping A: $X \to L$ where L is the support of an appropriate structure $\mathbf{L}$ of truth values (usually $L = [0, 1]$). In [5], Zadeh proposed a so called extension principle which became an important tool in fuzzy set theory and applications. The idea is that each function f: $X \to Y$ induces a corresponding function $\bar{f}$: $L^X \to L^Y$ (i.e., $\bar{f}$ is a function mapping fuzzy sets in X to fuzzy sets in Y) defined for each fuzzy set A in X by

$$\bar{f}(A)(y) = \bigvee_{x \in X,\ f(x) = y} A(x).$$

The function $\bar{f}$ is said to be obtained from f by the extension principle. Therefore, given a precise functional dependence expressed by f, one can

[1] Supported in part by the Project VS96037 of the MSMT CR and by IGS OU 1999.

0022-247X/00 $35.00

compute f (in fact, $\bar{f}$) even when the argument of f is described only approximately (and expressed by a fuzzy set). The aim of this note is to show that the function $\bar{f}$ preserves (in a natural way) similarity of fuzzy sets.

A general structure of truth values is represented by complete residuated lattices introduced into the context of fuzzy sets and fuzzy logic by Goguen [2] (see also [3, 4] for further justification from the point of view of logic). Recall that a complete residuated lattice is an algebra $\mathbf{L} = \langle L, \otimes, \to, \wedge, \vee, 0, 1 \rangle$ where (i) $\langle L, \wedge, \vee, 0, 1 \rangle$ is a complete lattice with the least element 0 and the greatest element 1, (ii) $\langle L, \otimes, 1 \rangle$ is a commutative monoid, (iii) $\otimes$ and $\to$ satisfy the adjointness property; i.e., $x \leq y \to z$ iff $x \otimes y \leq z$ holds. The most applied complete residuated lattices are those with $L = [0,1]$ and the following structures: Łukasiewicz ($a \otimes b = \max(a + b - 1, 0)$, $a \to b = \min(1 - a + b, 1)$), Gödel ($a \otimes b = \min(a, b)$, $a \to b = 1$ if $a \leq b$ and $= b$ otherwise), and product ($a \otimes b = a \cdot b$, $a \to b = 1$ if $a \leq b$ and $= b/a$ otherwise). For further properties of residuated lattices we refer to [2–4]. Note that if $L = [0,1]$ then $\otimes$ is a t-norm [3]. Moreover, each left-continuous t-norm $\otimes$ makes L into a complete residuated lattice putting $a \to b = \vee\{c \mid a \otimes c \leq b\}$.

2. RESULTS

We are going to show that the function $\bar{f}$ preserves similarity. Recall that a similarity relation in a set U is a binary fuzzy relation $E\colon U \times U \to L$ which is reflexive (i.e., $E(x, x) = 1$), symmetric (i.e., $E(x, y) = E(y, x)$), and transitive (i.e., $E(x, y) \otimes E(y, z) \leq E(x, z)$). There is a natural way to measure similarity between fuzzy sets. It is expressed in the following proposition (see, e.g., [1]).

PROPOSITION 2.1. *Let X be a set. The binary fuzzy relation E_X defined on L^X by $E_X(A_1, A_2) = \bigwedge_{x \in X} A_1(x) \leftrightarrow A_2(x)$ is a similarity relation in L^X.*

Here $\leftrightarrow$ denotes the biresiduum operation defined by $x \leftrightarrow y = (x \to y) \wedge (y \to x)$. Note that if interpreted on the linguistic level, two fuzzy sets are similar iff they contain the same elements.

In the case of the above noted Łukasiewicz, Gödel, and product t-norms we obtain

$$E(A_1, A_2) = \inf_{x \in X} 1 - |A_1(x) - A_2(x)| \qquad \text{(Łukasiewicz)}$$

$$E(A_1, A_2) = \inf_{x \in X} \min(A_1(x), A_2(x)) \qquad \text{(Gödel)}$$

$$E(A_1, A_2) = \inf_{x \in X} \min(A_1(x)/A_2(x), A_2(x)/A_1(x)) \qquad \text{(product)},$$

where we put $0/0 = 1$ and $a/0 = \infty$ for $a \neq 0$.

THEOREM 2.1. *Let $\bar{f}$ be the function obtained from f: $X \to Y$ by the extension principle. Then for every fuzzy set A_1, A_2 in X it holds that $E_X(A_1, A_2) \le E_Y(\bar{f}(A_1), \bar{f}(A_2))$.*

Proof. By the definition of $\bar{f}$ we have to show that

$$E_X(A_1, A_2) \le \bigwedge_{y \in Y} \left(\bigvee_{x \in X,\ f(x)=y} A_1(x) \leftrightarrow \bigvee_{x \in X,\ f(x)=y} A_2(x) \right)$$

which holds iff for each $y \in Y$ the following holds:

$$E_X(A_1, A_2) \le \bigvee_{x \in X,\ f(x)=y} A_1(x) \leftrightarrow \bigvee_{x \in X,\ f(x)=y} A_2(x).$$

To show this, we have to check that the left side of the inequality is less or equal to both $\bigvee_{x \in X,\ f(x)=y} A_1(x) \to \bigvee_{x \in X,\ f(x)=y} A_2(x)$ and $\bigvee_{x \in X,\ f(x)=y} A_2(x) \to \bigvee_{x \in X,\ f(x)=y} A_1(x)$. Due to symmetry we proceed for the first case only.

$$E_X(A_1, A_2) \le \bigvee_{x \in X,\ f(x)=y} A_1(x) \to \bigvee_{x \in X,\ f(x)=y} A_2(x)$$

is (by the adjointness property of residuated lattices) equivalent to

$$\left(\bigvee_{x \in X,\ f(x)=y} A_1(x) \right) \otimes E_X(A_1, A_2) \le \bigvee_{x \in X,\ f(x)=y} A_2(x)$$

which is true since

$$\begin{aligned} &\left(\bigvee_{x \in X,\ f(x)=y} A_1(x) \right) \otimes E_X(A_1, A_2) \\ &\quad = \bigvee_{x \in X,\ f(x)=y} (A_1(x) \otimes E_X(A_1, A_2)) \\ &\quad \le \bigvee_{x \in X,\ f(x)=y} (A_1(x) \otimes (A_1(x) \to A_2(x)) \le \bigvee_{x \in X,\ f(x)=y} A_2(x). \end{aligned}$$

The assertion is proved.

Note that the result says that the truth value of the proposition "if A_1 and A_2 are similar then $\bar{f}(A_1)$ and $\bar{f}(A_2)$ are similar" is 1 (full truth).

The following corollary is immediate.

COROLLARY 2.1. *Under the notation of Theorem 2.1, if f is an injective function (i.e., $x_1 \ne x_2$ implies $f(x_1) \ne f(x_2)$) then $E_X(A_1, A_2) = E_Y(\bar{f}(A_1), \bar{f}(A_2))$.*

The extension principle is sometimes formulated to extend a function

$$f\colon X_1 \times \cdots X_n \to Y \text{ into } \bar{f}\colon L^{X_1} \times \cdots L^{X_n} \to L^Y \text{ by}$$

$$\bar{f}(A_1, \ldots, A_n)(y) = \bigvee_{x_1 \in X_1, \ldots, x_n \in X_n, f(x_1, \ldots, x_n) = y} A_1(x_1) \otimes \cdots \otimes A_n(x_n) \quad (1)$$

for $A_1, \ldots, A_n$ being fuzzy sets in $X_1, \ldots, X_n$, respectively. For such a case we have the following assertion.

THEOREM 2.2. *Let $\bar{f}$ be the function obtained from $f\colon X_1 \times \cdots X_n \to Y$ by* (1). *Then for every fuzzy set $A_1, B_1 \in L^{X_1}, \ldots, A_n, B_n \in L^{X_n}$ it holds that*

$$E_{X_1}(A_1, B_1) \otimes \cdots \otimes E_{X_n}(A_n, B_n) \le E_Y\big(\bar{f}(A_1, \ldots, A_n), \bar{f}(B_1, \ldots, B_n)\big).$$

Proof. The assertion is a consequence of Theorem 2.1. Namely, putting $X = X_1 \times \cdots \times X_n$ and $A = A_1 \otimes \cdots \otimes A_n$, $B = B_1 \otimes \cdots \otimes B_n$, where $A \in L^X$ is defined by $A(x_1, \ldots, x_n) = A_1(x_1) \otimes \cdots \otimes A_n(x_n)$ (and similarly for B), Theorem 2.1 yields

$$E_X(A, B) \le E_Y\big(\bar{f}(A_1, \ldots, A_n), \bar{f}(B_1, \ldots, B_n)\big).$$

To prove the assertion, it is therefore sufficient to prove

$$E_{X_1}(A_1, B_1) \otimes \cdots \otimes E_{X_n}(A_n, B_n) \le E_X(A, B),$$

which holds iff for every $\mathbf{x} = \langle x_1, \ldots, x_n \rangle \in X_1 \times \cdots \times X_n$, $\mathbf{y} = \langle y_1, \ldots, y_n \rangle \in Y_1 \times \cdots \times Y_n$, the following holds:

$$E_{X_1}(A_1, B_1) \otimes \cdots \otimes E_{X_n}(A_n, B_n) \le A(\mathbf{x}) \to B(\mathbf{x}) \wedge B(\mathbf{y}) \to A(\mathbf{y}).$$

To check this inequality we verify

$$E_{X_1}(A_1, B_1) \otimes \cdots \otimes E_{X_n}(A_n, B_n) \le A(\mathbf{x}) \to B(\mathbf{x})$$

(due to symmetry we omit checking the second inequality which is to verify) which is equivalent to

$$A(\mathbf{x}) \otimes E_{X_1}(A_1, B_1) \otimes \cdots \otimes E_{X_n}(A_n, B_n) \le B(\mathbf{x}).$$

The last inequality is, indeed, true since

$$\begin{aligned} A(\mathbf{x}) &\otimes E_{X_1}(A_1, B_1) \otimes \cdots \otimes E_{X_n}(A_n, B_n) \\ &= A_1(x_1) \otimes E_{X_1}(A_1, B_1) \otimes \cdots \otimes A_n(x_n) \otimes E_{X_n}(A_n, B_n) \\ &\le A_1(x_1) \otimes (A_1(x_1) \to B_1(x_1)) \otimes \cdots \otimes A_n(x_n) \\ &\quad \otimes (A_n(x_n) \to B_n(x_n)) \\ &\le B_1(x_1) \otimes \cdots \otimes B_n(x_n) = B(\mathbf{x}). \end{aligned}$$

∎

The extension principle is used mainly in situations where no precise description of the input data is available, e.g., if a linguistic variable is used to describe the inputs. In such a case, using the extension principle we can make our model "tolerant to imprecision" of that kind. Our results say that if we do that, the exact description of the imprecise inputs (exact shape of membership function) does not matter.

ACKNOWLEDGMENTS

The paper was written during author's postdoc visit at the Center for Intelligent Systems, Binghamton University–SUNY, Binghamton, NY 13902. The support by the Center and by its director, Professor G. J. Klir, is gratefully acknowledged.

REFERENCES

1. R. Bělohlávek, Similarity relations in concept lattices, *J. Logic Comput.*, to appear.
2. J. A. Goguen, L-fuzzy sets, *J. Math. Anal. Appl.* **18** (1967), 145–174.
3. P. Hájek, "Metamathematics of Fuzzy Logic," Kluwer, Dordrecht, 1998.
4. U. Höhle, On the fundamentals of fuzzy set theory, *J. Math. Anal. Appl.* **201** (1996), 786–826.
5. L. A. Zadeh, The concept of a linguistic variable and its applications in approximate reasoning, *Inform. Sci.* **8** (1975), 199–251, 301–357; **9** (1975), 43–80.